MUMBAI UNIVERSITY

SEMESTER 1

APPLIED MATHEMATICS SOLVED PAPER – MAY 2018

N.B:- (1) Question no. 1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1(a) If $\tan\frac{x}{2} = \tanh\frac{u}{2}$, show that $u = \log[(\tan(\frac{\pi}{4}+\frac{x}{2}))]$ [3]

Ans : Given that : $\tan\frac{x}{2} = \tanh\frac{u}{2}$

$$\frac{u}{2} = \tanh^{-1}[\tan\frac{x}{2}]$$

$$\therefore u = 2\tanh^{-1}[\tan\frac{x}{2}]$$

By using Inverse hyperbolic function,

$$= \log\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

But $\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \frac{\frac{\pi}{4}+\tan\frac{x}{2}}{\frac{\pi}{4}-\tan\frac{x}{2}} = \tan(\frac{\pi}{4}+\frac{x}{2})$

$$\therefore u = \log[(\tan(\frac{\pi}{4}+\frac{x}{2}))]$$

Hence proved.

(b) Prove that the following matrix is orthogonal & hence find A^{-1}. [3]

$$A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Ans : Let $A = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}$

Transpose of A is given by ,

$$A^T = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$A.A^T = \frac{1}{9}\begin{bmatrix} -2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{I}{9}$$

The given matrix A is orthogonal.

The inverse of an orthogonal matrix is always equal to the Transpose of that particular matrix.

$$\therefore \quad A^{-1} = \frac{1}{3}\begin{bmatrix} -2 & 2 & 1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

(c) State Euler's theorem on homogeneous function of two variables & if u = $\frac{x+y}{x^2+y^2}$ then evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ **[3]**

Ans : Euler's theorem : If a function 'u' is homogeneous with degree 'n' then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$

Let u = $\frac{x+y}{x^2+y^2}$

Put x = xt and y = yt

$$F(x,y) = \frac{xt+yt}{(xt)^2+(yt)^2} = \frac{1}{t}\left[\frac{x+y}{x^2+y^2}\right]$$

$$= t^{-1}.f(u)$$

Hence the given function 'u' is homogeneous with degree n=-1

$$\therefore \ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$

$$\therefore \ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\left[\frac{x+y}{x^2+y^2}\right]$$

(d) If u = $r^2 cos\ 2\theta, v = r^2 sin\ 2\theta$. Find $\frac{\partial(u,v)}{\partial(r,\theta)}$. [3]

Ans : u = $r^2 cos\ 2\theta$ $v = r^2 sin\ 2\theta$

Diff. u and v w.r.t r and θ partially to apply it in jacobian

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} 2r cos\ 2\theta & -2r^2 sin\ 2\theta \\ 2r sin\ 2\theta & 2r^2 cos\ 2\theta \end{vmatrix}$$

$$= 4r^3 cos^2 2\theta + 4r^3 sin^2 2\theta$$

$$= 4r^3 (cos^2 2\theta + sin^2 2\theta)$$

$$\boxed{\frac{\partial(u,v)}{\partial(r,\theta)} = 4r^3}$$

(e) Find the nth derivative of cos 5x.cos 3x.cos x. [4]

Ans : let y = cos 5x.cos 3x.cos x

$$= \frac{cos\ (5x-3x)+cos\ (5x+3x)}{2}\ \text{cos x}\ \{ \text{cos A.cos B}=\tfrac{1}{2}[cos\ (A+B) + COS\ (A-B)]\}$$

$$= \frac{1}{2}\ [\ cos\ 2x.\ cos\ x + cos\ 8x.\ cos\ x\ ]$$

$$y = \frac{1}{4}\ [\ cos\ 3x + cos\ x + cos\ 9x + cos\ 7x\ ]$$

Take n th derivative,

$$n\ th\ derivative\ of\ cos\ (ax+b) = a^n cos\ \left(\frac{n\pi}{2} + ax + b\right)$$

$$\boxed{y_n = \frac{1}{4}\ [3^n cos\left(\frac{n\pi}{2} + 3x\right) + cos\ \left(\frac{n\pi}{2} + x\right) + 9^n cos\ \left(\frac{n\pi}{2} + 9x\right) + 7^n cos\ \left(\frac{n\pi}{2} + 7x\right)]}$$

(f) Evaluate : $\lim_{x \to 0} (\frac{2x+1}{x+1})^{\frac{1}{x}}$ [4]

Ans : let $L = \lim_{x \to 0} (\frac{2x+1}{x+1})^{\frac{1}{x}}$

Take log on both sides,

$\therefore \quad \log L = \lim_{x\to 0} \frac{1}{x}\left(\frac{2x+1}{x+1}\right)$

$= \lim_{x\to 0}\left(\frac{2x+1}{x^2+x}\right)$

Apply L'Hospital rule ,

$\therefore \quad \log L = \lim_{x\to 0}\left(\frac{2}{2x+1}\right)$

$\therefore \quad \log L = 2$

$\therefore \quad L = e^2$

Q. 2(a) Solve $x^4 - x^3 + x^2 - x + 1 = 0.$ [6]

Ans : $x^4 - x^3 + x^2 - x + 1 = 0$

Multiply the given eqn by (x+1),

$(x+1)(x^4 - x^3 + x^2 - x + 1)=0$

$x^5 = (-1)$

But $-1 = \cos\pi + i\sin\pi$

$\therefore \quad x = [\cos\pi + i\sin\pi]^{1/5}$

But By De Moivres theorem ,

$(\cos x + i\sin x)^n = \cos nx + i\sin nx$

$\therefore \quad x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

Add period $2k\pi$,

$\therefore \quad x = \cos(1+2k)\frac{\pi}{5} + i\sin(1+2k)\frac{\pi}{5}$

Where k = 0,1,2,3,4.

The roots of given eqn is given by ,

Put k=0 $\quad x_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5} = e^{\pi/5}$

k=1 $\quad x_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} = e^{3\pi/5}$

k=2 $x_2 = cos\frac{\pi}{1} + i\,sin\frac{\pi}{1} = e^{\pi/1}$

k=3 $x_3 = cos\frac{7\pi}{5} + i\,sin\frac{7\pi}{5} = e^{7\pi/5}$

k=4 $x_4 = cos\frac{9\pi}{5} + i\,sin\frac{9\pi}{5} = e^{9\pi/5}$

The roots of eqn are : $e^{\pi/5}, e^{3\pi/5}, e^{\pi/1}, e^{7\pi/5}, e^{9\pi/5}$.

(b) If y=$e^{\tan^{-1}x}$. Prove that

$$(1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0 \qquad [6]$$

Ans : y = $e^{\tan^{-1}x}$ ....(1)

Diff. w.r.t x ,

$y_1 = e^{\tan^{-1}x}\frac{1}{x^2+1}$

$(x^2+1)y_1 = e^{\tan^{-1}x} = y$ --------(from 1)

Again diff. w.r.t x,

$(x^2+1)y_2 + 2xy_1 = y_1$ ....(1)

Now take n th derivative by applying Leibnitz theorem,

Leibnitz theorem is :

$(uv)_n = u_n v + {}^{n}_{1}C u_{n-1}v_1 + {}^{n}_{2}C u_{n-2}v_2 + \cdots + uv_n$

$u = (x^2+1), v = y_2$ ...for first term in eqn (1)

$u = 2x, v = y_1$ ...for second term in eqn (1)

$\therefore (1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n - y_{n+1} = 0$

$\therefore (1+x^2)y_{n+2} + [2(n+1)x-1]y_{n+1} + n(n+1)y_n = 0$

Hence Proved.

(c) Examine the function $f(x, y) = xy(3 - x - y)$ for extreme values & find maximum and minimum values of $f(x, y)$. [8]

Ans : $$f(x, y) = xy(3 - x - y) = 3xy - x^2y - xy^2$$

Diff. function w.r.t x and y partially,

$\frac{\partial f(x,y)}{\partial x} = 3y - 2xy - y^2$ $\frac{\partial f(x,y)}{\partial y} = 3x - x^2 - 2xy$

$\frac{\partial f(x,y)}{\partial x} = 0$ $\frac{\partial f(x,y)}{\partial y} = 0$

$3y - 2xy - y^2 = 0$ & $3x - x^2 - 2xy = 0$

y=0 , 3-2x-y=0 & x=0 , 3-x-2y=0

Stationary points are : (0,0) ,(3,0) , (0,3) , (1,1)

r $= \frac{\partial^2 f}{\partial x^2} = -2y$, $t = \frac{\partial^2 f}{\partial y^2} = -2x$

s $= \frac{\partial^2 f}{\partial x \partial y} = 3 - 2x - 2y$

$s^2 = (3 - 2x - 2y)^2$

rt-s^2=4xy-$(3 - 2x - 2y)^2$

For point (0,0), rt-s^2 =-9 < 0

The point is of maxima .

For point (3,0), rt-$s^2 = -9 < 0$

The point is of maxima .

For (0,3), rt-s^2 = -9 < 0

The point is of maxima.

For point (1,1), rt-$s^2 = 3 > 0$

The point is of minima .

(a) **Maximum values : At (0,0) , (0,3), (3,0)**

At point <u>(0,0)</u> f(max)=0

At point (0,3) f(max)=0

At point (3,0) f(max)=0

(b) Minimum values : At (1,1)

At point (1,1) f(min)=1

The maximum and minimum values of function are 0 and 1.

Q.3(a) Investigate for what values of μ and λ the equation x+y+z=6; x+2y+3z=10; x+2y+λz=μ have

(i)no solution,

(ii) a unique solution,

(iii) infinite no. of solution. **[6]**

Ans : Given eqn : x+y+z=6 , x+2y+3z=10 , x+2y+λz=μ

A X = B

$$\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 2 & \lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

Argumented matrix is : $\left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\1 & 2 & 3 & 10\\1 & 2 & \lambda & \mu\end{array}\right]$

$R_2 - R_1,$

$$\rightarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 1 & \lambda-1 & \mu-6\end{array}\right]$$

$R_3 - R_2,$

$$\rightarrow \left[\begin{array}{ccc|c}1 & 1 & 1 & 6\\0 & 1 & 2 & 4\\0 & 1 & \lambda-3 & \mu-10\end{array}\right]$$

(i) When λ=3,$\mu \neq 10$ $then\ r(a) = 2, r(A \vdots B) = 3$

r(A)$\neq r(A \vdots B)$

Hence for λ=3 , $\mu \neq 10$ system is inconsistent.

No solution exist.

(ii) When $\lambda \neq 3, \mu \neq 10$, $r(A) = r(A \vdots B) = 3$
Unique solution exist.

(iii) When $\lambda = 3, \mu = 10$ $\quad r(A) = r(A \vdots B) = 2 < 3$
Infinite solution.

(b) If u = f($\frac{y-x}{xy}$, $\frac{z-x}{xz}$) , show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$ [6]

Ans : let u = f(r,s)

$\therefore$ r = $\frac{y-x}{xy}$ $\qquad$ $\therefore$ s = $\frac{z-x}{xz}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\frac{1}{x^2} + \frac{\partial u}{\partial s}(\frac{-1}{x^2})$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = \frac{\partial u}{\partial r}\frac{(-1)}{y^2} + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(\frac{1}{z^2})$$

$$\therefore \quad x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial s} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$\boxed{x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0}$$

Hence proved.

(c) Prove that log($\frac{a+ib}{a-ib}$)=2i$\tan^{-1}\frac{b}{a}$ & cos [i log($\frac{a+ib}{a-ib}$] = $\frac{a^2-b^2}{a^2+b^2}$ [8]

Ans : let L = log($\frac{a+ib}{a-ib}$)

Using logarithmic properties ,

L = log(a+ib) – log(a-ib)

$= \frac{1}{2}\log(a^2+b^2) + i.\tan^{-1}\frac{b}{a} - [\frac{1}{2}\log(a^2+b^2) - i.\tan^{-1}\frac{b}{a}]$

$$L = 2i\tan^{-1}\frac{b}{a}$$

$$\therefore \log\left(\frac{a+ib}{a-ib}\right) = 2i\tan^{-1}\frac{b}{a}$$

Hence Proved.

$$\therefore \frac{a+ib}{a-ib} = e^{2i\tan^{-1}\frac{b}{a}} = \cos\left(2\tan^{-1}\frac{b}{a}\right) + i\sin\left(2\tan^{-1}\frac{b}{a}\right)$$

$$\frac{a+ib}{a-ib}\times\frac{a+ib}{a+ib} = \cos\left(2\tan^{-1}\frac{b}{a}\right) + i\sin\left(2\tan^{-1}\frac{b}{a}\right)$$

$$\frac{a^2-b^2}{a^2+b^2} + imaginary = \cos\left(2\tan^{-1}\frac{b}{a}\right) + i\sin\left(2\tan^{-1}\frac{b}{a}\right)$$

Separate real and imaginary parts

$$\cos\left(2\tan^{-1}\frac{b}{a}\right) = \frac{a^2-b^2}{a^2+b^2}$$

From 1st result ,

$$\cos\left[i\log\left(\frac{a+ib}{a-ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$$

Hence Proved.

Q.4(a) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *Prove that*

$$x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-\sin u.\cos 2u}{4\cos^3 u}$$ **[6]**

Ans : $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Put x = xt and y = yt to find degree.

$$\therefore u=\sin^{-1}\left(\frac{xt+yt}{\sqrt{xt}+\sqrt{yt}}\right)$$

$$\therefore \sin u = t^{1/2}.\frac{x+y}{\sqrt{x}+\sqrt{y}} = t^{\frac{1}{2}}.f(x,y)$$

The function sin u is homogeneous with degree ½.

But sin u is the function of u and u is the function of x and y.

By Euler's theorem ,

$$xu_x + yu_y = G(u) = n.\frac{f(u)}{f'(u)} = \frac{1}{2}\text{tanu}$$

$$\therefore\ x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = G(u)[G'(u) - 1]$$

$$= \frac{1}{2}\tan u[\frac{sec^2u-2}{2}]$$

$$= \frac{1}{4}\tan u[\frac{tan^2u-1}{1}]$$

$$= \frac{1}{4} \times \frac{\sin u}{\cos u}[\frac{sin^2u-cos^2u}{cos^2u}]$$

$$\therefore\ x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{-sinu.cos2u}{4cos^3u}$$

Hence Proved.

(b) Using encoding matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ encode and decode the message "ALL IS WELL" . [6]

Ans : Let encoding matix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

The message is ALL IS WELL and Let B is the matrix in number form,

$$B = \begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

The encoded message is given by,

13 12 12 0 28 19 23 23 17 12 12 0

"MLL ASWWQLL "

Inverse of encoding matix A = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is given by ,

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

Decoded matrix is given by ,

$$B = A^{-1}.C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \begin{bmatrix} 13 & 12 & 28 & 23 & 17 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 12 & 9 & 0 & 5 & 12 \\ 12 & 0 & 19 & 23 & 12 & 0 \end{bmatrix}$$

(c) Solve the following equation by Gauss Seidal method :

$$10x_1 + x_2 + x_3 = 12$$

$$2x_1 + 10x_2 + x_3 = 13$$

$$2x_1 + 2x_2 + 10x_3 = 14$$

[8]

Ans : By Gauss Seidal method ,

Given eqn : $10x_1 + x_2 + x_3 = 12$

$2x_1 + 10x_2 + x_3 = 13$

$2x_1 + 2x_2 + 10x_3 = 14$

From given eqn : $|10|>|1|+|1|$

$|10|>|2|+|1|$

$|10|>|2|+|2|$

The given eqn are in correct order.

$$\therefore \ x_1 = \frac{1}{10}[12 - x_2 - x_3]$$

$$\therefore \ x_2 = \frac{1}{10}[13 - 2x_1 - x_3]$$

$$\therefore\ x_3 = \frac{1}{10}[14 - 2x_2 - 2x_1]$$

I) **For 1st iteration :** take $x_2 = 0, x_3 = 0$

$$x_1 = \frac{1}{10}[12] = 1.2$$

$x_1 = 1.2\ , x_3 = 0$ gives $x_2 = 1.06$

$x_1 = 1.2, x_2 = 1.06$ gives $x_3 = 0.948$

II) **For 2nd iteration :** take $x_2 = 1.06, x_3 = 0.948$

$$x_1 = \frac{1}{10}[12 - 1.06 - 0.948] = 0.9992$$

$x_1 = 0.992, x_3 = 0.948$ gives $x_2 = 1.0068$

$x_1 = 0.992, x_2 = 1.0068$ gives $x_3 = 1.0002$

III) **For 3rd iteration :** $x_2 = 1.0068, x_3 = 1.0002$

$$x_1 = \frac{1}{10}[12 - 1.0068 - 1.0002] = 0.9993$$

$x_1 = 0.993, x_3 = 1.0002$ gives $x_2 = 1.00$

$x_1 = 0.993, x_2 = 1.00$ gives $x_3 = 1.00$

Result : $x_1 = 1.00, x_2 = 1.00, x_3 = 1.00$

Q.5(a) If u = $e^{xyz} f\left(\frac{xy}{z}\right)$ $where\ f(\frac{xy}{z})$ is an arbitrary function of $\frac{xy}{z}$.

Prove that : $x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz.u$ **[6]**

Ans: let $\frac{xy}{z} = w$ $\therefore$ u = $e^{xyz}.f(w)$

Diff. u w.r.t. x partially,

$$\frac{\partial u}{\partial x} = e^{xyz} f'(w) + f(w).e^{xyz}.yz$$

Diff. u w.r.t y partially ,

$$\frac{\partial u}{\partial y} = e^{xyz} f'(w) + f(w).e^{xyz}.xz$$

Diff. u w.r.t y partially ,

$$\frac{\partial u}{\partial z} = e^{xyz} f'(w) + f(w).e^{xyz}.xy$$

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = xe^{xyz} f'(w) + f(w).e^{xyz}.xyz + ze^{xyz} f'(w) + f(w).e^{xyz}.xyz \quad ...(1)$$

$$y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = ye^{xyz} f'(w) + f(w).e^{xyz}.xyz + ze^{xyz} f'(w) + f(w).e^{xyz}.xyz \quad ...(2)$$

From (1) and (2),

$$x\frac{\partial u}{\partial x} + z\frac{\partial u}{\partial z} = y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2xyz.u$$

Hence Proved.

(b) Prove that $sin^5\theta = \frac{1}{16}[sin\,5\theta - 5sin\,3\theta + 10sin\,\theta]$ **[6]**

Ans : let $x = \cos\theta + isin\ \theta$ $\quad \frac{1}{x} = \cos\theta - isin\ \theta$

$$2\cos\theta = x + \frac{1}{x} \qquad sin\ \theta = \frac{1}{2i}(x - \frac{1}{x})$$

For sin θ $take\ fifth\ power\ on\ both\ sides$,

$$sin^5\theta = [\frac{1}{2i}(x - \frac{1}{x})]^5 = \frac{1}{32i}[x^5 - \frac{1}{x^5} - 5\left(x^3 - \frac{1}{x^3}\right) + 10(x^1 - \frac{1}{x^1})]$$

But $x^n = cos\,n\theta + isin\,n\theta$, $x^{-n} = cos\,n\theta - isin\,n\theta$

$$x^n - x^{-n} = 2isin\,n\theta$$

$$\therefore \quad sin^5\theta = \frac{1}{32i}[\,2isin\,5\theta - 5 \times 2isin\,3\theta + 10 \times 2isin\,\theta\,]$$

$$\therefore \quad sin^5\theta = \frac{1}{16}[sin\,5\theta - 5sin\,3\theta + 10sin\,\theta]$$

(c) i) Prove that log(sec x)=$\frac{1}{2}x^2 + \frac{1}{12}x^4 + \cdots$

ii) Expand $2x^3 + 7x^2 + x - 1$ $in\ powers\ of\ x - 2.$ **[8]**

Ans : (i) Let E = log (sec x)

= - log (cos x)

= - log [1- ($\frac{x^2}{2!} - \frac{x^4}{4!}$)]

= $-[-\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) - \frac{1}{2}\left(\frac{x^2}{2!} - \frac{x^4}{4!}\right) + \cdots]$

$$E = log(sec\, x) = \frac{1}{2}x^2 + \frac{1}{12}x^4 + ..$$

(ii) let f(x)= $2x^3 + 7x^2 + x - 1$

Here a = 2

$f(x) = 2x^3 + 7x^2 + x - 1$ $\quad$ $f(2) = 45$

$f'(x) = 6x^2 + 14x + 1$ $\quad$ $f'(2) = 53$

$f''(x) = 12x + 14$ $\quad$ $f''(2) = 38$

$f'''(x) = f'''(2) = 12$

Taylor's series is :

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$

$$2x^3 + 7x^2 + x - 1 = 45 + (x-2)53 + \frac{(x-2)^2}{2!}38 + \frac{(x-a)^3}{3!}12$$

$$2x^3 + 7x^2 + x - 1 = 45 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Q.6(a) Prove that $\sin^{-1}(cosec\, \theta) = \frac{\pi}{2} + i.log(cot\, \frac{\theta}{2})$ **[6]**

Ans : we have to prove this $\sin^{-1}(cosec\, \theta) = \frac{\pi}{2} + i.log(cot\, \frac{\theta}{2})$

$$(cosec\, \theta) = sin[\frac{\pi}{2} + i.log\left(cot\, \frac{\theta}{2}\right)]$$

R.H.S = $sin[\frac{\pi}{2} + i.log\left(cot\, \frac{\theta}{2}\right)]$

= $cos\,[i.log\left(cot\, \frac{\theta}{2}\right)]$ ………{ sin ($\frac{\pi}{2} + x$)=cos x }

= $cos\, hlog\left(cot\, \frac{\theta}{2}\right)$ ……….{ cos ix = cos hx }

$$= \frac{1}{2}[e^{log\left(cot\frac{\theta}{2}\right)} + e^{-log\left(cot\frac{\theta}{2}\right)}]$$ ....{ cos hx = $\frac{1}{2}[e^{x} + e^{-x}]$ }

$$= \frac{1}{2}[cot\ \frac{\theta}{2} + \frac{1}{cot\frac{\theta}{2}}]$$

$$= \frac{1}{2}tan\ \frac{\theta}{2}[\ 1 + cot^2\frac{\theta}{2}]$$

$$= \frac{1}{2}tan\ \frac{\theta}{2}[\frac{sin^2\frac{\theta}{2}+cos^2\frac{\theta}{2}}{sin^2\frac{\theta}{2}}]$$ ....{ $\tan\theta = \frac{\sin\theta}{\cos\theta}$ = $\frac{1}{cot\ \theta}$ }

$$= \frac{1}{2} \times \frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}} \times \frac{1}{sin^2\frac{\theta}{2}}$$

$$= \frac{1}{sin\ \theta}$$

$$= cosec\ \theta \quad = \quad L.H.S$$

$$\therefore (cosec\ \theta) = sin[\frac{\pi}{2} + i.log\left(cot\ \frac{\theta}{2}\right)$$

$$\therefore \sin^{-1}(cosec\ \theta) = \frac{\pi}{2} + i.log(cot\ \frac{\theta}{2})$$ **Hence Proved.**

(b) Find non singular matrices P and Q such that A=$\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}$ [6]

Ans : A = $\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix}$

For PAQ form ,

A = $I_{3\times3} . A_{3\times4} . I_{3\times3}$

$$\begin{bmatrix}1 & 2 & 3 & 2\\ 2 & 3 & 5 & 1\\ 1 & 3 & 4 & 5\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$R_2 - 2R_1, R_3 - R_1,$

$$\begin{bmatrix}1&2&3&2\\0&-1&-1&-3\\0&-1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_2 - 2C_1, C_3 - 3C_1, C_4 - 2C_1,$

$$\begin{bmatrix}1&0&0&0\\0&-1&-1&-3\\0&-1&1&3\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-1&0&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$R_3 + R_2,$

$$\begin{bmatrix}1&0&0&0\\0&-1&-1&-3\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-3&-2\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$C_3 - C_2, C_4 - 3C_2,$

$$\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$-R_2,$

$$\left[\begin{array}{cc|cc}1&0&0&0\\0&1&0&0\\\hline 0&0&0&0\end{array}\right]=\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}A\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Now A is in Normal form .

Compare this w.r.t A=PAQ form ,

$\therefore$ **P=** $\begin{bmatrix}1&0&0\\2&-1&0\\-3&1&1\end{bmatrix}$ **Q =** $\begin{bmatrix}1&-2&-1&4\\0&1&-1&-3\\0&0&1&0\\0&0&0&1\end{bmatrix}$

$\therefore$ Rank of given matrix A is 2.

(c) Obtain the root of $x^3 - x - 1 = 0$ by Regula Falsi Method

(Take three iteration). [8]

Ans : Equation : $x^3 - x - 1 = 0$

$\therefore \; f(x) = x^3 - x - 1$

$f(0) = -1 < 0 \;\; and \;\; f(1) = -1 < 0 \; and \;\; f(2) = 5 > 0.$

Root of given eqn lies between 1 and 2.

(x_0, y_0) = (1,-1) (x_1, y_1) = (2,5)

$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ = 1.2249

$f(x_2) = -0.3871 < 0$

Next iteration :

(x_0, y_0) = (1.2249,-0.3871)

(x_1, y_1) = (1.667,1.9654)

$\therefore \;\; x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ =1.2976

$f(x_2) = -0.1127 < 0$

Next iteration :

(x_0, y_0) = (1.2976,-0.1127)

(x_1, y_2) = (1.667,1.9654)

$x_2 = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} = 1.3176$

The root of given eqn after 3rd iteration is 1.3176.